

Photon dispersion in causal sets

Jeffrey D. Scargle

NASA Ames Research Center, Moffett Field, CA 94035-1000

Slobodan N. Simić

Department of Mathematics, San José State University, San José, CA 95192-0103

A very small dispersion in the speed of light may be observable in Fermi time- and energy-tagged data on variable sources, such as gamma-ray bursts (GRB) and active galactic nuclei (AGN). We describe a method to compute the size of this effect by applying the Feynman sum-over-histories formalism for relativistic quantum electrodynamics to a discrete model of space-time called *causal set theory*.

I. INTRODUCTION

The vacuum speed of light may be a function of photon energy. Even a very small such dependence might yield energy-dependent time delays which, over the long journeys from distant astronomical sources, may accumulate to a level measurable by the Fermi Gamma Ray Space Telescope [1]. The expectation from general considerations [2] is that the scale of the effect is linear in the photon's energy:

$$\frac{dt}{T} \sim \frac{dE}{E_{\text{Planck}}}, \quad (1)$$

where the Planck energy is

$$E_{\text{Planck}} = \sqrt{\frac{\hbar c^5}{G}} \sim 1.22 \times 10^{19} \text{ GeV}. \quad (2)$$

This relation gives delays over astronomical distances of seconds (GeV energies) to hours (TeV energies).

A motivation for this suggestion is the possibility that space-time may be lumpy, fuzzy, or even discrete, on small scales, and that this lumpiness may affect photon propagation in a dispersive way. We adopt the extreme viewpoint provided by *causal set theory*, an approach to quantum gravity pioneered by Rafael Sorkin [13], with the founding principle that space-time is a *partially ordered* set of purely discrete points. The partial order nicely reflects the causal relationships between events fundamental to special relativity. We also adopt the Feynman sum-over-histories quantum electrodynamics, which is greatly simplified by the postulated discreteness: path integrals, and the difficulties associated with divergences and defining the measure on the space of all paths are replaced by discrete sums.

II. CAUSAL SETS

Causal set (CS) theory postulates that at very small scales – below those of ordinary experience, and even

below the smallest scales accessible to particle physics experiments – space-time consists of a set of discrete points. The usual space-time continuum is thus regarded as a macroscopic construct that does not exist. One only requires that the large scale limit of the CS recovers known, macroscopic, continuum physics.

In mathematical terms, a *causal set* is a pair (\mathcal{C}, \prec) , where \mathcal{C} is a set and \prec is a binary relation on \mathcal{C} satisfying the following properties (x, y, z etc. are general points in \mathcal{C}):

1. **Partial Order:** Some, but not all, pairs x, y of points are ordered: $x \prec y$;
2. **Transitivity:** If $x \prec y$ and $y \prec z$, then $x \prec z$;
3. **Anti-reflexivity:** No point is related to itself: $x \not\prec x$;
4. **Local finiteness:** For all $x, y \in \mathcal{C}$, the set $[x, y] = \{z \in \mathcal{C} : x \prec z \text{ and } z \prec y\}$ is finite.

The first two items are consistent with, and are taken to express, the causal relations of special relativity: $x \prec y$ (read “ x precedes y ”) means that y is in the forward light-cone of x , and can affect x . If separated by a space-like interval x and y are not ordered. Transitivity here simply expresses the relation between nested light cones.

One consequence of the above ansatz is that all space-time information is contained in the causal connections among the points. For example, any topological or metric notions must be extracted from the discrete (i.e., combinatorial) structure of (\mathcal{C}, \prec) [6, 7, 10, 11], without any reference to a continuum. This also means that the action (see below) for the step $x \rightarrow y$ can depend on only the causal relation between the two points, plus the energy and polarization of the photon. We make use of this simplifying fact below.

Note that arbitrarily defined partial orderings among the points in general can yield causal sets that cannot be realized as a subset of any Lorentzian manifold. The *sprinkling* procedure [5] for generating CS

order relations was invented to avoid these microscopic inconsistencies, to avoid other problems that might foul up the macroscopic continuum limit[14] mentioned above, and to ensure a kind of statistical Lorentz invariance. The process begins by randomly sprinkling points in a continuous Lorentzian space-time manifold M , of the appropriate dimension. For example, to model flat space, the points would be uniformly and independently distributed[15] in M . The fundamental density ϱ of this distribution is frequently taken to be such that the mean (continuum) separation of points is the Planck scale. Then the partial order relations among the sprinkled points are derived from the causal (light-cone) relations assessed in the continuum M . Here we adopt this sprinkling concept as defining causal sets, which in the literature are said to be *embedded* into the continuum M .

III. PHOTON PROPAGATION: FEYNMAN SUMS OVER SPACE-TIME PATHS

Given a causal set (\mathcal{C}, \prec) embedded in some $M = \mathbb{R}^{d+1}$, where d is the spatial dimension and there is one time dimension, we consider photon propagation in \mathcal{C} . Given spacetime points x, y , one asks for the value of the *propagator* $\mathbb{K}(x, y)$, *i.e.* the quantum-mechanical amplitude for a photon emitted at x to be detected at y ; the probability of this transtion is $|\mathbb{K}(x, y)|^2$. The standard sum over paths formalism[3, 4] states that the amplitude for each path from x to y is the product of the amplitudes of each step comprising the path, and the amplitude for the process is obtained by summing these amplitudes over **all paths** from x to y .

It is more realistic to take \mathcal{S} , a subset of \mathcal{C} , to represent an unresolved astronomical source, and another subset \mathcal{D} to represent the detector. Then the usual prescription is that the amplitude for the process of interest – photons emitted from the source arriving at the detector, symbolically $\mathcal{S} \rightarrow \mathcal{D}$ – is the sum of the amplitudes over all paths starting beginning at some point in \mathcal{S} and ending at some point in \mathcal{D} . The probability of the process $\mathcal{S} \rightarrow \mathcal{D}$ is the square of the complex amplitude of the process. This prescription is valid if and only if one is not able to determine which point in \mathcal{D} the photon arrives at, nor from which point in \mathcal{S} the photon originates. If the detector were to be divided into *pixels*, such that the identity of the pixel at which the photon arrives is knowable, then as usual amplitudes are additive within pixels, but probabilities are additive across pixels.

To compute the propagator $\mathbb{K}(x, y)$ we use the Feynman path integral approach which asserts that the propagator is the “integral” of $\exp(iS(\gamma)/\hbar)$ over the

space of all continuous paths γ from x to y , where $S(\gamma)$ is the action associated with γ . Whereas in the continuous case this integral is not well-defined in a rigorous sense and its meaning, definition and computation are still an area of investigation, in the causal set scenario the integral reduces to a sum, to which it is easier to give a precise meaning. Related work can be found in [8, 9], but unlike these references our approach takes the *all paths* spirit of [4] quite literally, allowing *causal, non-causal, and superluminal paths* and relying on the phase averages for large action paths as appropriate weights.

All that remains is specification of the action $S(x, y)$ for an elementary step $x \rightarrow y$, for arbitrary points x and y in \mathcal{C} , for the corresponding amplitude is [3]

$$S(x, y) = S_0 e^{i \frac{S(x, y)}{\hbar}} \quad (3)$$

where \hbar is Planck’s constant, and S_0 is a normalization factor which can be determined by considering the operator for an infinitesimal advance in time ([3], §6).

IV. MATHEMATICAL RESULTS

To effect a suitable choice of the action just described, take the amplitude of each causal or non-causal jump to be a or b , respectively, where a and b are suitably chosen complex numbers. Given a path $\gamma = x_0 x_1 \cdots x_n$ consisting of n steps, set $S(\gamma) = a^{n-k} b^k$, where k is the number of non-causal steps in γ , and $n - k$ the number of causal ones. The propagator from $x \equiv x_0$ to $y \equiv x_n$ is then

$$\mathbb{K}(x, y) = \sum_{n=1}^{\infty} \sum_{k=0}^n a^{n-k} b^k p_{n,k}(x, y), \quad (4)$$

where $p_{n,k}(x, y)$ denotes the number of paths of length n from x to y with exactly k non-causal steps. We now adopt a statistical view and, in a slight abuse of notation, treat \mathbb{K} and $p_{n,k}(x, y)$ as averages over random sprinklings.

One obvious problem with this definition is that for $n \geq 2$ and $k \geq 1$, $p_{n,k}(x, y)$ is infinite. To rectify this problem, we pick a sufficiently large number $r > 0$ and allow jumps (both casual and non-causal) only up to distance r . That is, $\gamma = x_0 x_1 \cdots x_n$ is defined to be an *admissible path* only if each step satisfies $d(x_j, x_{j+1}) \leq r$, where d is the distance metric. This distance should be defined in the CS sense (*e.g.* as in [11]), but in this preliminary study we have ignored this nuance. If we denote the corresponding propagator by \mathbb{K}_r , then

$$\mathbb{K}_r(x, y) = \sum_{n=1}^{\infty} \sum_{k=0}^n a^{n-k} b^k p_{n,k}^r(x, y), \quad (5)$$

where $p_{n,k}^r(x, y)$ is the expected number of admissible paths of length n from x to y with exactly k non-causal steps [we will call such paths (n, k) -paths]. One can think of the sum in equation (5) either as an ordinary series (whose convergence will be assured if $|a|$ and $|b|$ are sufficiently small) or as an *oscillatory* series, whose convergence is defined differently. We choose the former interpretation of equation (5).

To compute $\mathbb{K}_r(x, y)$ for any $x, y \in \mathbb{R}^{d+1}$, we need a suitable expression for $p_{n,k}^r(x, y)$. The amplitudes of causal and non-causal steps will be represented by defining two functions $\nu_r, \mu_r : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ as follows:

$$\nu_r(z) = \begin{cases} 1 & \text{if } z \prec \mathbf{0} \text{ and } d(z, \mathbf{0}) \leq r \\ 0 & \text{otherwise,} \end{cases} \quad (6)$$

and

$$\mu_r(z) = \begin{cases} 1 & \text{if } z \not\prec \mathbf{0} \text{ and } d(z, \mathbf{0}) \leq r \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

See Figure 1. Observe that $\nu_r(x - y) = 1$ if and only if there is an admissible non-causal jump (i.e., path of length one) from x to y , and $\mu_r(x - y) = 1$ if and only if there is an admissible non-causal jump from x to y . It is not hard to see that $\nu_r = \mathbf{1}_{B_r} \mathbf{1}_{L_-}$ and $\mu_r = \mathbf{1}_{B_r} \mathbf{1}_{L_+}$ where $\mathbf{1}_{B_r}$ is the characteristic function of the ball B_r of radius r centered at the origin, and

$\mathbf{1}_{L_\pm}$ are the characteristic functions of the future and past light cones of the origin. See Figure 2.

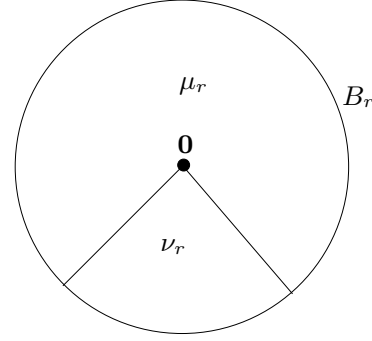


FIG. 1: The region marked ν_r is the backward light cone of $\mathbf{0}$, truncated at distance r . The remaining region marked μ_r includes the forward light cone and the remaining region with space-like separations from, and therefore causally disconnected from, $\mathbf{0}$.

It is possible to derive explicit formulas for $p_{n,k}^r(x, y)$ using convolution-type integrals of functions ν_r, μ_r . For lack of space, we do not present these formulas. It can then be shown that the functions $p_{n,k}^r$ satisfy the following family of integral equations:

$$p_{n+1,k}^r(x, y) = \varrho \int_{\mathbb{R}^{d+1}} p_{n,k}^r(x, z) \nu_r(z - y) dz + \varrho \int_{\mathbb{R}^{d+1}} p_{n,k-1}^r(x, z) \mu_r(z - y) dz, \quad (8)$$

for all $n \geq 1$ and $1 \leq k \leq n$. A lengthy calculation yields the following integral equation for the propagator:

$$\begin{aligned} a\varrho \int_{\mathbb{R}^{d+1}} K_r(x, z) \nu_r(z - y) dz + b\varrho \int_{\mathbb{R}^{d+1}} \mathbb{K}_r(x, z) \mu_r(z - y) dz &= \mathbb{K}_r(x, y) - a\nu_r(x - y) - b\mu_r(x - y) \\ &- \sum_{n=2}^{\infty} b^n p_{n,n}^r(x, y) + b\varrho \sum_{n=1}^{\infty} b^n \int_{\mathbb{R}^{d+1}} p_{n,n}^r(x, z) \mu_r(z - y) dz. \end{aligned} \quad (9)$$

To solve equation (9), we observe that both $\mathbb{K}_r(x, y)$ and $p_{n,n}^r(x, y)$ are translation invariant. This means that $\mathbb{K}_r(x, y) = \Psi_r(x - y)$ and $p_{n,n}^r(x, y) = P_n^r(x - y)$, for some functions $\Psi_r, P_n^r \in L^1(\mathbb{R}^{d+1})$. Note also that the integrals in equation (9) are convolutions. Applying the Fourier transform $\hat{\cdot}$, solving for $\hat{\Psi}_r$, and applying the inverse Fourier transform yields:

$$\Psi_r = \sum_{n=0}^{\infty} \varrho^n (a\nu_r + b\mu_r)^{*(n+1)}, \quad (10)$$

where $(a\nu_r + b\mu_r)^{*(n+1)}$ denotes the $(n+1)$ -st convo-

lution power of $a\nu_r + b\mu_r$. To compute the geometric convolution sum in equation (10), we need to recall that the space $L^1(\mathbb{R}^{d+1})$ of Lebesgue integrable functions is a Banach algebra [12] with respect to multiplication given by convolution $*$, but that it does not possess a unit element. However, a unity – usually denoted by δ (one can think of it as the Dirac delta “function”) – can be adjoined to $L^1(\mathbb{R}^{d+1})$ to obtain a new Banach algebra $\tilde{L}^1(\mathbb{R}^{d+1})$ with unity (cf., [12], §10.1). Then the sum in equation (10) becomes

$$\Psi_r = (a\nu_r + b\mu_r) * \{\delta - \varrho(a\nu_r + b\mu_r)\}^{-1}, \quad (11)$$

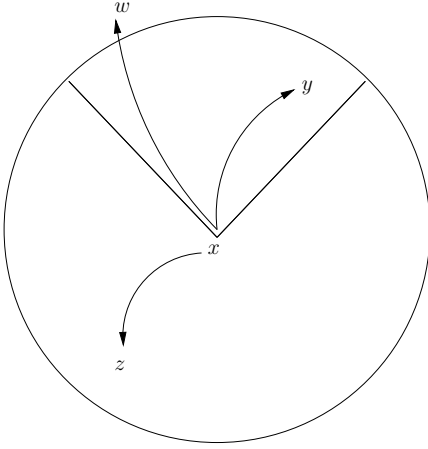


FIG. 2: Jumps from x to y and x to z are admissible, but not from x to w .

where the inverse is taken in the extended algebra $\tilde{L}^1(\mathbb{R}^{d+1})$.

Now let \mathcal{S} and \mathcal{D} be subsets of \mathbb{R}^{d+1} representing the source and the detector (see Section III). Then the probability a photon traveling from \mathcal{S} to \mathcal{D} is

$$\text{Prob}(\mathcal{S} \rightarrow \mathcal{D}) = \left| \int_{\mathcal{S} \times \mathcal{D}} \mathbb{K}_r(x, y) dx dy \right|^2. \quad (12)$$

We leave the discussion of this integral for various \mathcal{S} and \mathcal{D} to our forthcoming longer paper.

V. NUMERICAL RESULTS

A matrix formalism simplifies numerical evaluation of the propagators for small CSs. For a causal set consisting of N points $x_n, n = 1, 2, \dots, N$, define the amplitude matrix

$$\mathbb{A} = [\mathbb{A}_{n,m}] = [A(x_n, x_m)], \quad (13)$$

generalizing the connection matrix of graph theory by replacing the 1 – 0 indicator of connectivity with the full amplitude from eq. (3) for the step between the points. The product of \mathbb{A} with itself,

$$[\mathbb{A}^2]_{n,m} = \sum_j A_{n,j} A_{j,m}, \quad (14)$$

gives the amplitude for two-step transitions $n \rightarrow m$ with a single intermediate stop at j . Similarly $[\mathbb{A}^k]_{n,m}$ gives the amplitude of the same transition with $k - 1$ intermediate stops, and the series

$$\mathbb{Q} = \mathbb{A} + \mathbb{A}^2 + \mathbb{A}^3 + \dots \quad (15)$$

gives the amplitude summed over all paths. As long as $\|\mathbb{A}\| < 1$ for some norm, this series converges to the computationally useful form

$$\mathbb{Q} = (\mathbb{I} - \mathbb{A})^{-1} - \mathbb{I}, \quad (16)$$

where \mathbb{I} is the identity matrix.

So the amplitude of the transition $\mathcal{S} \rightarrow \mathcal{D}$ is the sum of the elements of \mathbb{Q} in the sub-block with column indices labeling points in \mathcal{S} and row indices labeling points in \mathcal{D} , and the final probability for $\mathcal{S} \rightarrow \mathcal{D}$ is the absolute square of this sum.

Considering this result and the comments about pixels which identify detected photons made above, we see how precisely and simply these matrix operations implement the fundamental sum-over-histories quantum rules: multiply amplitudes of steps on a path; add amplitudes of all paths between indistinguishable starting and final states; add probabilities corresponding to distinguishable final states.

After this construct was developed during a CAMCOS project at San José State University we discovered that Steven Johnston has a similar approach [8].

The formalism developed here seems to be a promising approach to computing photon dispersion in the context of causal set theory. A future publication will elaborate the analytical approach in Section IV and the numerical approach in Section V, including results on the speed of light as a function of energy.

Acknowledgments

The authors wish to thank David Mattingly, Steven Johnston, Robert Israel, Harald Hanche-Olsen, and Rafael Sorkin for valuable comments, and Ricky Fernandez and students in the CAMCOS program at SJSU for various contributions.

-
- [1] Abdo, A. A. et al. (2009), A limit on the variation of the speed of light arising from quantum gravity effects, *Nature*, **462**, 331-334.
 [2] Amelino-Camelia, G. Lee Smolin, L. (2009), “Prospects for constraining quantum gravity disper-

sion with near term observations,” *Phys. Rev. D* **80:084017**, arXiv:0906.3731v3

- [3] Feynman, R. P. (1948), “Space-Time Approach to Non-Relativistic Quantum Mechanics,” *Reviews of Modern Physics*, **20**, 367-387.

- [4] Feynman, R. P. (1985), “QED: the strange theory of light and matter,” Princeton University Press.
- [5] Henson, J., The causal set approach to quantum gravity, [arXiv:gr-qc/0601121](#),
- [6] Henson, J. (2006) “Constructing an interval of Minkowski space from a causal set,” *Classical and Quantum Gravity*, **23**, L29-35 [arXiv:gr-qc/0601121](#)
- [7] Ilie, R., Thompson, G. B. and Reid, D. (2006) “A numerical study of the correspondence between paths in a causal set and geodesics in the continuum,” *Class.Quant.Grav.* **23**, 3275-3286 [arXiv:gr-qc/0512073v1](#)
- [8] Johnston, S. (2008) “Particle propagators on discrete spacetime,” *Class. Quantum Grav.*, **25**, 202001 [arXiv:0806.3083v2](#)
- [9] Johnston, S. (2009) “Feynman Propagator for a Free Scalar Field on a Causal Set,” *Physical Review Letters*, **103**, 180401. [arXiv:0909.0944v2](#)
- [10] Major, S., Rideout, D., and Surya, S. (2007) “On Recovering Continuum Topology from a Causal Set,” *J. Math. Phys.*, **48**, 032501, [arXiv:gr-qc/0604124v3](#)
- [11] Rideout, D., and Wallden, P. (2009) “Spacelike distance from discrete causal order,” *Class.Quant.Grav.*, **26**, 155013, [arXiv:0810.1768v3](#) [[gr-qc](#)]
- [12] Walter Rudin, *Functional analysis*, second ed., McGraw Hill, 1991.
- [13] Sorkin, R. First Steps with Causal Sets, Proceedings of the Ninth Italian Conference of the same name, held Capri, Italy, September, 1990. General Relativity and Gravitational Physics, eds. R. Cianci, R. de Ritis, M. Francaviglia, G. Marmo, C. Rubano, P. Scudellaro.
- [14] The viewpoint here is that the CS is the reality, and the continuum is the approximation to it; this is the opposite of the usual view.
- [15] This means that the probability of sprinkling n points is a region of volume V is $(\rho V)^n e^{-\rho V}/n!$. Hence this is often called a Poisson process, but the essence is the independent and uniform nature of the random distribution, not the Poisson distribution function of counts in cells.